The Coulomb interaction and the inverse Faddeev-Popov operator in QCD

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(Dated: February 1, 2008)

We give a proof of a local relation between the inverse Faddeev-Popov operator and the non-Abelian Coulomb interaction between color charges.

When the QCD Hamiltonian is expressed entirely in terms of gauge-invariant variables, a nonlocal operator, $\Gamma^{ab}(\mathbf{y},\mathbf{x})$, appears in the role of the non-Abelian analog of $(8\pi|\mathbf{y}-\mathbf{x}|)^{-1}$, the 'static' interaction between electric charges in Coulomb-gauge QED. $\Gamma^{ab}(\mathbf{y},\mathbf{x})$ has the form

$$\Gamma^{ab}(\mathbf{y},\mathbf{x}) = -\tfrac{1}{2}\mathcal{C}^{ab}(\mathbf{y},\mathbf{x}) \tag{1}$$

with

$$\mathcal{C}^{ab}(\mathbf{y},\mathbf{x}) = \int d\mathbf{r}\,\mathcal{D}^{aq}(\mathbf{y},\mathbf{r})\partial^2_{(\mathbf{r})}\mathcal{D}^{qb}(\mathbf{r},\mathbf{x})\,, \tag{2}$$

where $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$ is the inverse Faddeev-Popov operator. Evaluating the inverse Faddeev-Popov operator $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$, by calculating its expectation value in a particular state vector or by some other means, is important for determining the boundaries of the regions within which it is bounded. [1, 2, 3] Evaluating $\Gamma^{ab}(\mathbf{y},\mathbf{x})$ is necessary for calculating the forces between colored objects, such as those between heavy static quarks. The conjecture that the unboundedness of $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$ as $|\mathbf{y}-\mathbf{x}|\to\infty$ is related to the unbounded growth of the force between color-bearing objects, and thereby to color-confinement, [1, 2, 3] suggests a close relationship between points at which $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$ and $\Gamma^{ab}(\mathbf{y},\mathbf{x})$ become unbounded.

In addition to the *nonlocal* relation expressed in Eq. (2), there is also a *local* relation between $\Gamma^{ab}(\mathbf{y},\mathbf{x})$ and $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$,

$$\mathcal{C}^{ab}(\mathbf{y},\mathbf{x}) = \frac{\partial\left(g\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})\right)}{\partial g}\,, \tag{3}$$

which, to the best of our knowledge, first appeared in a paper by Swift. [4] Because Eq. (3) expresses $\Gamma^{ab}(\mathbf{y},\mathbf{x})$ as a local functional of $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$, it makes the relation between the infrared behavior of $\Gamma^{ab}(\mathbf{y},\mathbf{x})$ and that of $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$ much more transparent.

The inverse Faddeev-Popov operator $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$ is defined by the relation

$$\partial\cdot D^{ca}_{(\mathbf{y})}\,\mathcal{D}^{ab}(\mathbf{y},\mathbf{x}) = \delta_{cb}\delta(\mathbf{y}-\mathbf{x}) \tag{4}$$

where

$$\partial\cdot D^{ab}_{(\mathbf{x})} = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}\delta_{ab} + gf^{aqb}A^q_i(\mathbf{x})\right)\,; \tag{5}$$

$A^q_i(\mathbf{x})$ represents a transverse gauge field. It can, for example, be the gauge field in the Coulomb gauge; or it might be the gauge-invariant field $A^q_{\mathsf{Gl}\,i}$ constructed within the Weyl ($A_0 = 0$) gauge, [5] which has been identified with the Coulomb-gauge field. [6] $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$ can be represented as the series

$$\mathcal{D}^{ab}(\mathbf{y},\mathbf{x}) = \sum_{n=0}^{\infty}\mathcal{D}^{ab}_{(n)}(\mathbf{y},\mathbf{x}) \tag{6}$$

with

$$\begin{aligned}\mathcal{D}^{ab}_{(n)}(\mathbf{y},\mathbf{x}) \quad &= g^n f^{\vec{\alpha}ab}_{(n)}\int\frac{d\mathbf{z}(1)}{4\pi|\mathbf{y}-\mathbf{z}(1)|}A^{\alpha_1}_{l_1}(\mathbf{z}(1))\frac{\partial}{\partial z(1)_{l_1}}\int\frac{d\mathbf{z}(2)}{4\pi|\mathbf{z}(1)-\mathbf{z}(2)|}\times\\ &A^{\alpha_2}_{l_2}(\mathbf{z}(2))\frac{\partial}{\partial z(2)_{l_2}}\cdots\int\frac{d\mathbf{z}(n)}{4\pi|\mathbf{z}(n{-}1)-\mathbf{z}(n)|}A^{\alpha_n}_{l_n}(\mathbf{z}(n))\frac{\partial}{\partial z(n)_{l_n}}\frac{1}{4\pi|\mathbf{z}(n)-\mathbf{x}|}\,;\end{aligned} \tag{7}$$

$f^{\vec{\alpha}ab}_{(n)}$ represents the chain of SU(N) structure constants

$$f^{\vec{\alpha}bh}_{(n)} = f^{\alpha_1 b u_1}\; f^{u_1\alpha_2 u_2}\; f^{u_2\alpha_3 u_3}\; \dots\; f^{u_{(n-2)}\alpha_{(n-1)}u_{(n-1)}} f^{u_{(n-1)}\alpha_n h}\,, \tag{8}$$

where repeated superscripted indices are summed; the chain reduces for $n = 1$ to $f^{\vec{\alpha}bh}_{(1)} = f^{\alpha bh}$; and for $n = 0$, to $f^{\vec{\alpha}bh}_{(0)} = -\delta_{bh}$. These properties of $f^{\vec{\alpha}ab}_{(n)}$ enable us to conclude that, for $n = 0$ and $n = 1$, the respective expressions for $\mathcal{D}^{ab}_{(n)}(\mathbf{y},\mathbf{x})$ are

$$\mathcal{D}^{ab}_{(0)}(\mathbf{y},\mathbf{x}) = \frac{-\delta_{ab}}{4\pi|\mathbf{y}-\mathbf{x}|} \tag{9}$$

and

$$\mathcal{D}^{ab}_{(1)}(\mathbf{y},\mathbf{x}) = gf^{\delta ab}\int\frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|}A^{\delta}_{k}(\mathbf{z})\frac{\partial}{\partial z_k}\left(\frac{1}{4\pi|\mathbf{z}-\mathbf{x}|}\right)\,. \tag{10}$$

In Ref. [6], we pointed out that $\mathcal{D}^{\,ab}(\mathbf{y},\mathbf{x})$ obeys the integral equation [7]

$$\mathcal{D}^{\,ab}(\mathbf{y},\mathbf{x}) = -\left(\frac{\delta_{ab}}{4\pi|\mathbf{y}-\mathbf{x}|} + gf^{\delta au}\int\frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|}A^{\delta}_{k}(\mathbf{z})\frac{\partial}{\partial z_k}\mathcal{D}^{\,ub}(\mathbf{z},\mathbf{x})\right)\,. \tag{11}$$

Eq. (11) can also be obtained from the defining equation for $\mathcal{D}^{\,ab}(\mathbf{y},\mathbf{x})$,

$$\left(\delta_{ah}\partial^2_{(\mathbf{z})} + gf^{aqh}A^{q}_{i}(\mathbf{z})\partial^{(\mathbf{z})}_{i}\right)\mathcal{D}^{hb}(\mathbf{z},\mathbf{x}) = \delta_{ab}\delta(\mathbf{z}-\mathbf{x}) \tag{12}$$

by integrating both sides of the equation, as shown by

$$-\int\frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|}\left\{\left(\delta_{ah}\partial^2_{(\mathbf{z})} + gf^{aqh}A^{q}_{i}(\mathbf{z})\partial^{(\mathbf{z})}_{i}\right)\mathcal{D}^{hb}(\mathbf{z},\mathbf{x})\right\} = \frac{-\,\delta_{ab}}{4\pi|\mathbf{y}-\mathbf{x}|}\,. \tag{13}$$

As was pointed out in Ref. [4], it is possible to represent $\mathcal{C}^{ab}(\mathbf{y},\mathbf{x})$ as the series

$$\mathcal{C}^{ab}(\mathbf{y},\mathbf{x}) = \sum_{n=0}^{\infty}\mathcal{C}^{ab}_{(n)}(\mathbf{y},\mathbf{x}) \tag{14}$$

and to observe, from iterating Eq. (11), that, order by order, each order examined confirms the relation

$$\mathcal{C}^{ab}_{(n)}(\mathbf{y},\mathbf{x}) = \frac{d}{dg}\,\left(g\mathcal{D}^{ab}_{(n)}(\mathbf{y},\mathbf{x})\right)\,. \tag{15}$$

Ref. [4] then points out that this fact can be used to prove Eq. (3). We will give a complete proof of Eq. (3) that does not require a perturbative decomposition of $\mathcal{C}^{ab}(\mathbf{y},\mathbf{x})$. We use Eq. (11) to represent $\mathcal{D}^{\,bh}(\mathbf{y},\mathbf{r})$, multiply both sides of that equation by $\partial^2_{(\mathbf{r})}\mathcal{D}^{qb}(\mathbf{r},\mathbf{x})$, and integrate over $\mathbf{r}$, to obtain

$$\mathcal{C}^{ab}(\mathbf{y},\mathbf{x}) = \mathcal{D}^{ab}(\mathbf{y},\mathbf{x}) - gf^{\delta au}\int\frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|}A^{\delta}_{\mathsf{GI}\,k}(\mathbf{z})\frac{\partial}{\partial z_k}\mathcal{C}^{ub}(\mathbf{z},\mathbf{x})\,. \tag{16}$$

We then *define* $\bar{\mathsf{C}}^{ab}(\mathbf{y},\mathbf{x})$ as

$$\bar{\mathsf{C}}^{ab}(\mathbf{y},\mathbf{x}) = \frac{\partial\left(g\mathcal{D}^{\,ab}(\mathbf{y},\mathbf{x})\right)}{\partial g}\,. \tag{17}$$

and apply the operation of multiplying by g and then differentiating with respect to g to both sides of Eq. (11), obtaining

$$\bar{\mathsf{C}}^{ab}(\mathbf{y},\mathbf{x}) = -\frac{\delta_{ab}}{4\pi|\mathbf{y}-\mathbf{x}|}\overbrace{-f^{\delta au}\int\frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|}A^{\delta}_{k}(\mathbf{z})\frac{\partial}{\partial z_k}\sum_{n=0}^{\infty}(n+2)g^{n+1}\mathsf{d}^{\,ub}_{(n)}(\mathbf{z},\mathbf{x})}^{\mathsf{R}_2}\,, \tag{18}$$

where we use Eq. (7) to write $\mathcal{D}^{ab}_{(n)}(\mathbf{y},\mathbf{x}) = g^n \mathsf{d}^{ab}_{(n)}(\mathbf{y},\mathbf{x})$ and where $\mathsf{d}^{ab}_{(n)}(\mathbf{y},\mathbf{x})$ is independent of g. We write the second term on the right-hand side of Eq. (18)

$$\mathsf{R}_2 = \mathsf{R}_2(A) + \mathsf{R}_2(B) \tag{19}$$

with

$$\mathsf{R}_2(A) = -gf^{\delta au}\int \frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|} A^{\delta}_k(\mathbf{z}) \frac{\partial}{\partial z_k} \sum_{n=0}^{\infty} g^n \mathsf{d}^{ub}_{(n)}(\mathbf{z},\mathbf{x}) = -gf^{\delta au}\int \frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|} A^{\delta}_k(\mathbf{z}) \frac{\partial}{\partial z_k} \mathcal{D}^{ub}(\mathbf{z},\mathbf{x}) \tag{20}$$

and

$$\mathsf{R}_2(B) = -gf^{\delta au}\int \frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|} A^{\delta}_k(\mathbf{z}) \frac{\partial}{\partial z_k} \sum_{n=0}^{\infty} (n+1) g^n \mathsf{d}^{ub}_{(n)}(\mathbf{z},\mathbf{x}) = -gf^{\delta au}\int \frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|} A^{\delta}_k(\mathbf{z}) \frac{\partial}{\partial z_k} \bar{\mathsf{C}}^{ub}(\mathbf{z},\mathbf{x})\,. \tag{21}$$

Since

$$-\frac{\delta_{ab}}{4\pi|\mathbf{y}-\mathbf{x}|} + \mathsf{R}_2(A) = \mathcal{D}^{ab}(\mathbf{y},\mathbf{x}), \tag{22}$$

it follows that

$$\bar{\mathsf{C}}^{ab}(\mathbf{y},\mathbf{x}) = \mathcal{D}^{ab}(\mathbf{y},\mathbf{x}) - gf^{\delta au}\int \frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|} A^{\delta}_{\mathsf{GI}\,k}(\mathbf{z}) \frac{\partial}{\partial z_k} \bar{\mathsf{C}}^{ub}(\mathbf{z},\mathbf{x})\,. \tag{23}$$

Since Eqs. (16) and (23) are identical, and both are linear integral equations, $\mathcal{C}^{ab}(\mathbf{y},\mathbf{x})$ and $\bar{\mathsf{C}}^{ab}(\mathbf{y},\mathbf{x})$ are identical as well, and Eq. (3) is proven. We note, also, that the fact that $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x}) = \mathcal{D}^{ba}(\mathbf{x},\mathbf{y})$, [6] implies that $\Gamma^{ab}(\mathbf{y},\mathbf{x}) = \Gamma^{ba}(\mathbf{x},\mathbf{y})$.

An interesting consequence of this theorem is the proper generalization, to non-Abelian gauge theories, of the static potential between charges in Abelian, Coulomb-gauge QED,

$$\int d\mathbf{x}d\mathbf{y}\rho(\mathbf{x}) \frac{1}{8\pi|\mathbf{y}-\mathbf{x}|}\rho(\mathbf{y}) \equiv -\frac{1}{2}\int d\mathbf{x}\rho(\mathbf{x})\left(\frac{1}{\partial^2}\right)\rho(\mathbf{x})\,. \tag{24}$$

We might, perhaps, wonder whether one could extend Eq. (24) to non-Abelian theories by replacing the Laplacian operator in Eq. (24) with the Faddeev-Popov operator $\partial\cdot D$. But Eq. (3) informs us that this 'naive' substitution is not allowed. The proper extension of Eq. (24) into the non-Abelian domain is to write the non-Abelian nonlocal interaction between color-charges symbolically as

$$\int d\mathbf{x}\left(j^a_0(\mathbf{x}) + J^{a\,\mathsf{T}\,\dagger}_{0\,(\mathsf{GI})}(\mathbf{x})\right)\left\{\frac{\partial\{g[(\partial\cdot D)^{-1}]^{ab}\}}{\partial g}\right\}\left(j^b_0(\mathbf{x}) + J^{b\,\mathsf{T}}_{0\,(\mathsf{GI})}(\mathbf{x})\right) \tag{25}$$

where $\partial\cdot D$ is given by Eq. (5).

Eq. (3) has significant advantages over Eq. (2). For a fixed set of points $\mathbf{y}$ and $\mathbf{x}$, Eq. (2) expresses Γ as a *nonlocal* functional of $\mathcal{D}$, so that it is not very intuitive that the behavior of $\Gamma^{ab}(\mathbf{y},\mathbf{x})$ as $|\mathbf{y}-\mathbf{x}|\to\infty$ is related to the behavior of $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$ as $|\mathbf{y}-\mathbf{x}|\to\infty$. In contrast, Eq. (3) expresses Γ as a *local* functional of $\mathcal{D}$ and the relation between the infrared behavior of $\Gamma^{ab}(\mathbf{y},\mathbf{x})$ and that of $\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$ becomes more transparent. Moreover, as illustrated in the work of Szczepaniak and Swanson, [8], Eq. (3) enables one to eliminate an integration over one spatial variable in evaluating expectation values of the Hamiltonian for trial wave functions that represent the physical QCD vacuum.

The authors thank Prof. E. S. Swanson for calling their attention to Refs. [4] and [8] after their initial preprint had been posted. The research of KH was supported by the Department of Energy under Grant No. DE-FG02-92ER40716.00 and that of HCR was supported by the Department of Energy under Grant No. DE-FG02-91ER40651-TASKB.

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